Commutators in Division Algebras

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Contents

A Question!

A Brief Review

- Consider a special property P in a ring:
  - Commutativity
  - Algebraicity

Whether one can specify a set or a substructure S, such that the property P for S implies the property P for the whole ring.

S=Set of Generators

- **Question 1:**

Let A be an algebraic structure generated by set S. Whether property P for S implies property P for A?

- **Question 2:**

Let A be an algebraic structure generated by set S. Whether property P for S implies property Q for A?

Candidates for S

- **General Division Algebras**
- **Multiplicative and additive commutators**
- **Subgroups D' and $[D, D]$.**
- **Normal subgroups of D^*.**
- **Division Algebras with Involutions**
- **Symmetric elements**
- **Skew-symmetric elements**
- **Unitary elements.**

Some Basic Definitions

Let D be a division algebra over its center F.

Definition

- Denote by D' the multiplicative subgroup of D^* generated by the all multiplicative commutators of D.

Definition

- Denote by $[D, D]$ the additive subgroup of commutators generated by the all additive commutators of D.

Definition

- Denote by $T(D)$ the vector space generated by the all multiplicative commutators over F.

Some Definitions

Let D be division ring with center F.

Definition

- We say A is radical over B if for every element $a \in A$ there exists integer $n = n(a)$ such that $a^n \in B$.

Definition

- Element $a \in A$ is called periodic if there exists integer n such that $a^n = 1$.

Definition

- Element $a \in A$ is called algebraic of degree n over center if satisfies a polynomial $f(x) \in F[x]$ of degree n.

Commutators in Division Rings

Commutators in Division Rings

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Their Generating Role

Commutators as Generators

"first course in non-commutative rings" due to T.Y. LAM

Theorem (Corollary 13.19, p. 211)

- Let D be a non-commutative division ring with center F. Then D is generated as an F-algebra by all additive commutators of D.

Theorem (Corollary 13.9, p. 207)

- A non-commutative division ring D is generated as a division ring by all of its multiplicative commutators.

Conjecture (M.A, Akbari-Ariannejad-Madadi)

- A division ring D with center F is generated as a vector space over F by all of its multiplicative commutators.

$T(D) = D$!

Theorem (M.A., Akbari-Ariannejad-Madadi)

- If D is algebraic with characteristic zero, then $T(D) = D$.

Trace Functions

- Let K/F be a field extension with $dim_F K = n$. For $a \in K$, define

$$L_a : K \longrightarrow K,$$

where $L_a(b) = ab$.

Definition

- The Trace function is defined for all $a \in K$ by

$$T_{K/F}(a) = Tr(L_a).$$

Trace Functions

Theorem

- **Let K/F be a field extension with $dim_F K = n$ and**

$$f(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

be the minimal polynomial of $a \in K$. Then

$$T_{K/F}(a) = -\frac{n}{m} b_{m-1}.$$

Wedderburn's Theorem

Theorem (Wedderburn)

- Let D be a division ring with center F

$a \in D^*$ be algebraic with minimal polynomial $f(x) \in F[x]$ of degree n.

Then

$$f(x) = (x - a_1) \dots (x - a_n) \in D[x].$$

Remark

- Note that linear factors are not unique!

Trace Formula

- Let $a \in D^*$ be algebraic with minimal polynomial

$$f(x) = (x-a)\dots(x-a_{n-1}) \in D[x].$$

Then

$$\begin{aligned} T_{F(a)/F}(a) &= a + a_1 + \cdots + a_{n-1} \\ &= a + d_1 a d_1^{-1} + \dots d_{n-1} a d_{n-1}^{-1} \\ &= a(1 + a^{-1} d_1 a d_1^{-1} + a^{-1} d_2 a d_2^{-1} + \\ &\quad \cdots + a^{-1} d_{n-1} a d_{n-1}^{-1}) \\ &= ad, \end{aligned}$$

where $d \in F(a) \cap T(D)$.

Theorem (M.A., Akbari-Ariannejad-Madadi)

- Let $a \in D$ be algebraic and $T_{F(a)/F}(a) \neq 0$, then $a^{-1} \in T(D)$.

$T(D)$ as Lie Ideal

Theorem (M. Aaghabli)

- Let D be an algebraic non-commutaitve division ring with center F. Then $T(D)$ is a non-central Lie ideal of D.

Theorem (M. Aaghabali)

- Let D be a centrally finite division ring over F. Then $T(D) = D$.

Finite Dimension $T(D)$

Theorem (M.A., Akbari-Ariannejad-Madadi)

- **Let D be a division ring with center F. If $dim_F T(D) = n < \infty$, then $dim_F D < \infty$.**

Commutators in Division Rings

Commutators in Division Rings

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Commutativity Conditions

Two Important Commutativity Conditions

Theorem (Wedderburn's Little Theorem)

- Every finite division ring is commutative.

Theorem (Kaplansky)

- If D is a division ring radical over its center, then D is commutative.

Finiteness Conditions

Theorem (Herstein-Procesi-Schacher)

- If D is a division ring with center F whose all additive commutators are radical over the center, then

$$dim_F D \leq 4$$

Commutativity Conditions

Conjecture (Herstein)

- Every division ring whose all multiplicative commutators are radical over its center must be commutative.

- General case is still open!

- Herstein (1978): Statement holds when commutators are periodic.

- Herstein (1978): Statement holds for centrally finite division rings.

$$dim_F D = n^2 \leq \infty.$$

Commutativity Conditions

- Herstein (1980): Statement holds for division rings with uncountable centers.

- Putcha-Yaqub (1974): The conjecture is true if the radical degree is a power of 2.

- Mahdavi-Akbari (1996): The conjecture is true if the radical degree is a power of 6.

Herstein Conjecture (Special Case)

Theorem (Mahdavi (1995))

Let D be an algebraic division algebra over its center F. If D' is radical over the center, then D is commutative.

Theorem (Mahdavi (1995))

Let D be a division algebra over its center F. If D' is radical over the center, then D is commutative.

Theorem (M.A., Akbari-Ariannejad-Madadi)

Let D be a division algebra over its center F. If $T(D)$ is radical over the center, then D is commutative.

Jacobson Theorem

Theorem (Jacobson)

- Every division algebra algebraic over a finite field is commutative.

Theorem (Mahdavi (1996))

- Every division algebra whose multiplicative group of commutators is algebraic over a finite field is commutative.

Noether-Jacobson Theorem

Theorem (Noether-Jacobson)

- Every non-commutative algebraic division ring over its center contains a non-central separable element.

Theorem (Mahdavi (1995))

- Every non-commutative algebraic division ring over its center contains a non-central separable element in its multiplicative subgroup of commutators.

Commutators in Division Rings

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Algebraicity Conditions

Algebraic Division Algebras

A Brief Review

- Let D be a division ring:

- Multiplicative Commutators
- Additive Commutators
- Subgroups D' and $[D, D]$

Whether one can deduce algebraicity of D over center if mentioned sets and structures are algebraic over the center.

Algebraicity of D' and $[D, D]$

Theorem (Akbari-Mahdavi (1996))

- Let D' be algebraic over the center, then D is algebraic over the center.

Theorem (Akbari-Ariannejad-Mehraabaadi (1998))

- Let $[D, D]$ be algebraic over the center, then D is algebraic over the center, provided $char(D) = 0$.

Algebraic commutators

Theorem (M.A., Akbari-Ariannejad-Madadi)

- Let D be a division algebra over its center F. If all multiplicative commutators are algebraic over F, then D is algebraic provided that F is UNCOUNTABLE.

- Assume $a \in D \setminus F$ and consider $y \in D^*$ arbitrarily.
- Either $y \in C_D(a)$ or $y \notin C_D(a)$.
- $y \notin C_D(a)$, for every $r \in F$ we have:

$$0 \neq b = ay - ya = a(y+r) - (y+r)a = (a(y+r)a^{-1}(y+r)^{-1} - 1)(y+r)a$$

- For every $r \in F, (y+r)ab^{-1}$ is algebraic over F.

$$f(t) \in F[t]; \;\; f((y+r)ab^{-1}) = 0$$

- put $c = ab^{-1}$, then

$$((y+r)c)^n + \alpha_1((y+r)c)^{n-1} + \cdots + \alpha_{n-1}((y+r)c) + \alpha_n = 0$$

Algebraic commutators

- $(y+r)(c((y+r)c)^{n-1}+\alpha_1 c((y+r)c)^{n-2})+\cdots+\alpha_{n-1}c)=-\alpha_n$

- $-\alpha_n(y+r)^{-1}=c((y+r)c)^{n-1}+\alpha_1 c((y+r)c)^{n-2})+\cdots+\alpha_{n-1}c$

- **assume the set of all words of finite length consisting of two letters** y, c.
- **consider vector space generated by the set of all such words.**
- **clearly, for every** $r\in F$, **we have** $(y+r)^{-1}\in W$.
- $dim_F W$ **is countable but** F **is uncountable**

Algebraic commutators

- hence we could find that $(y + r_1)^{-1}, \ldots, (y + r_m)^{-1}$ are linearly dependent over F.

Theorem

- Let D be a division algebra and K be a subfield of D. For $a \in D$, if $dim_K K[a] \geq n$, then for any distinct elements $\alpha_1, \ldots, \alpha_n \in Z(D)$, $(a - \alpha_1)^{-1}, \ldots, (a - \alpha_n)^{-1}$ are linearly independent.

- thus y is algebraic.
- now, assume $y \in C_D(a)$ and $z \notin C_D(a)$.
- $(y + r)z \notin C_D(a)$, for every $r \in F$ is algebraic.
- repeating argument for $(y + r)z$ we find that y is algebraic over F.

Algebraic commutators

Lemma (M.A., Akbari-Ariannejad-Madadi)

- Let D be a division ring with center F, $T(D)$ be Algebraic over F and $Char(D) = 0$. Then for any two Algebraic elements $a, b \in D$, the set $S = \{a + b, aba, a^2b\}$ is Algebraic over F.

Theorem (M.A., Akbari-Ariannejad-Madadi)

- Let D be a division ring with center F and $Char(D) = 0$. Then $T(D)$ is Algebraic over F if and only if D is Algebraic over F.

Algebraic commutators

Theorem (Jacobson)

- Every division ring whose elements are algebraic of bounded degree over its center is centrally finite.

Theorem (Bell-Drensky-Sharifi (2013))

- Every division ring whose elements are left algebraic of bounded degree over a not necessarily central subfield is centrally finite.

Algebraic commutators

Theorem (M.A., Akbari-Bien)

- Let D be a division ring with infinite center. If D contains element a such that $xax^{-1}a^{-1}$, for every $x \in D^*$ are left algebraic of bounded degree over a not necessarily central subfield, then D is centrally finite.

Algebraic commutators

Theorem (M.A., Akbari-Bien)

- Let D be a division ring with infinite center F and not necessarily central subfield K. If D contains a non-central normal subgroup N left algebraic of bounded degree n over K, then D is centrally finite.

Algebraic commutators

Theorem (M.A., Akbari-Bien)

- Let D be a division ring with infinite center and not necessarily central subfield K. Assume that K contains a non-central algebraic element a over the center. If all additive commutators $ax - xa$, for every $x \in D$ are left algebraic of bounded degree over K, then D is centrally finite.

Algebraic commutators

Theorem (M.A., Akbari-Bien)

- Let D be a division ring with center F and not necessarily central subfield K. Assume that D' is left algebraic of bounded degree over a K, then D is centrally finite.

Thank you for your attention